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2008 J. Phys. A: Math. Theor. 41 455205

(http://iopscience.iop.org/1751-8121/41/45/455205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 455205 (7pp) doi:10.1088/1751-8113/41/45/455205

Closed-form summations of certain hypergeometric-type series containing the digamma function

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Received 27 May 2008, in final form 17 September 2008
Published 15 October 2008

Abstract

Recently, interesting novel summation formulae for hypergeometric-type series containing a digamma function as a factor have been established by Miller (2006 *J. Phys. A: Math. Gen.* **39** 3011–20) mainly by exploiting already known results and using certain transformation and reduction formulae in the theory of the Kampé de Fériet double hypergeometric function. It is shown here that these, and several other series of this type, can be closed-form summed by simpler and more direct arguments based only on a derivative formula for the Pochhammer symbol and the theory of the digamma (or ψ) function and generalized hypergeometric function. In addition, a new reduction formula for the Kampé de Fériet function $F^{1:2;1}_{1:1;0}[z, z]$ is obtained.

PACS numbers: 02.30.−f, 02.30.Gp, 02.30.Lt
Mathematics Subject Classification: 33B15, 33C20, 65B10, 33C15

1. Introduction

A number of series involving the ψ (or digamma) function, given as the logarithmic derivative of the familiar gamma function $\Gamma(z)$ (see [1, p 258, equation (6.3.1)], [2, p 13, equation (1)] and [3])

$$\psi(z) = \frac{\mathrm{d}\log\Gamma(z)}{\mathrm{d}z} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad (z \in \mathbb{C}\backslash\mathbb{Z}_0^-;\, \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}),$$

can be found in the book ‘A table of series and products’ by Hansen [4, section 55, pp 360–6] (see also [5, 6], [7, pp 595–6] and [8, pp 151–3 and 160]).

Recently, several interesting novel summation formulae for hypergeometric-type series containing a digamma function as a factor have been established by Miller [9] mainly by exploiting already known results and using certain transformation and reduction formulae in the theory of the Kampé de Fériet double hypergeometric function [10–12]. Sums of this type occur often in mathematical physics and other applied areas especially when deriving asymptotic expansions and exact results for Mellin–Barnes and other integrals [13, 14].

In this sequel to the work of Miller, by simple arguments based only on the theory of the digamma function, a derivative formula for the Pochhammer symbol (see equations (1.7) and (1.8) below) and the theory of the generalized hypergeometric function ${}_pF_q$ (see section 2), we provide simple and short direct derivations of the following closed-form summations in terms of ${}_pF_q$:

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,(\lambda)_n\frac{z^n}{n!}=\frac{z}{(1-z)^{\lambda}}\,{}_2F_1\left[\begin{matrix}1,1;\\2;\end{matrix}\;z\right]\qquad(|z|<1),\tag{1.1}$$

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,z^n=\frac{1}{\lambda}\frac{z}{1-z}\,{}_2F_1\left[\begin{matrix}1,\quad\lambda;\\\lambda+1;\end{matrix}\;z\right]\qquad(|z|<1),\tag{1.2}$$

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,\frac{z^n}{n!}=\frac{1}{\lambda}z\,\mathrm{e}^{z}\,{}_2F_2\left[\begin{matrix}1,\quad 1;\\2,\lambda+1;\end{matrix}\;-z\right],\tag{1.3}$$

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,\frac{z^n}{(\lambda)_n}=\frac{1}{\lambda^2}z\,\mathrm{e}^{z}\,{}_2F_2\left[\begin{matrix}\lambda,\quad\lambda;\\\lambda+1,\lambda+1;\end{matrix}\;-z\right],\tag{1.4}$$

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,\frac{(\alpha)_n}{(\lambda)_n}z^n$$
$$=\frac{\alpha z}{\lambda^2(1-z)^{\alpha+1}}\,{}_3F_2\left[\begin{matrix}\alpha+1,\quad\lambda,\quad\lambda;\\\lambda+1,\lambda+1;\end{matrix}\;\frac{z}{z-1}\right]\qquad(\alpha\in\mathbb{C};\,|z|<1),\tag{1.5}$$

and in terms of the modified Bessel function of first kind $I_\nu(z)$ (for more details see, for instance, [1, p 374, section 9.6]):

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,\frac{z^n}{(\lambda)_n n!}$$
$$=z^{\frac{1-\lambda}{2}}\Gamma(\lambda)I_{\lambda-1}(2\sqrt{z})[\log\sqrt{z}-\psi(\lambda)]-z^{\frac{1-\lambda}{2}}\Gamma(\lambda)\frac{\partial}{\partial\lambda}I_{\lambda-1}(2\sqrt{z}),\tag{1.6}$$

which are valid for any complex λ, $\lambda\notin\mathbb{Z}_0^-$.

It is important to note that the factor $\psi(\lambda+n)-\psi(\lambda)$ which appears in the series (1.1)–(1.6) could be expressed as follows:

$$\psi(\lambda+n)-\psi(\lambda)=\frac{1}{(\lambda)_n}\frac{\partial}{\partial\lambda}\{(\lambda)_n\}:=\frac{1}{(\lambda)_n}\frac{\partial}{\partial\lambda}\left\{\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}\right\}\qquad(n\in\mathbb{N}_0:=\{0,1,2,\ldots\}),\tag{1.7}$$

where $(\alpha)_n$ stands for the Pochhammer symbol (or the *shifted factorial*, since $(1)_n=n!$) defined (for $\alpha\in\mathbb{C}$) by ([1, p 256, equation (6.1.23)] and [2, p 2])

$$(\alpha)_n=\begin{cases}1 & (n=0;\,\alpha\neq 0)\\\alpha(\alpha+1)\cdots(\alpha+n-1) & (n\in\mathbb{N})\end{cases}=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.\tag{1.8}$$

As a matter of fact, the derivatives of the Pochhammer symbol $(\lambda)_n$ and of $\frac{1}{(\lambda)_n}$ are, respectively, involved in (1.1) as well as in (1.4)–(1.6).

2. Summation of the series (1.1)–(1.6)

Throughout the text ${}_pF_q$ is the generalized hypergeometric function with p numerator and q denominator parameters, where p and q are nonnegative integers, which is, as usual, defined by means of the hypergeometric series (see [2, p 52] and [15, chapter 7])

$${}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},$$

whenever this series converges and elsewhere by analytic continuation. Here $(\alpha)_n$ denotes the Pochhammer symbol defined as in (1.8).

In general, the variable z (known as the argument), the numerator parameters $\alpha_1,\cdots,\alpha_p$ and the denominator parameters $\beta_1,\cdots,\beta_q$ take on complex values, provided that no denominator parameters are allowed to be zero or a negative integer. The ${}_pF_q$ function is symmetric in its numerator parameters, and likewise in its denominator parameters.

The series defining ${}_pF_q$ converges for all values of z when $p\leqslant q$. If $p=q+1$, the series converges when $|z|<1$, when $z=1$ if $\Re(\beta_1+\cdots+\beta_q-\alpha_1-\cdots-\alpha_p)>0$ and when $z=-1$ if $\Re(\beta_1+\cdots+\beta_q-\alpha_1-\cdots-\alpha_p)>-1$.

In this section, we first deduce the summation formula given by (1.1). To this end, upon using (1.7) and the following familiar binomial expansion [15, p 453, entry (7.3.1.1)]

$$\frac{1}{(1-z)^{\lambda}}=\sum_{n=0}^{\infty}(\lambda)_n\frac{z^n}{n!}\qquad(|z|<1),$$

we obtain

$$\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,(\lambda)_n\frac{z^n}{n!}=\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}(\lambda)_n\frac{z^n}{n!}\right\}=\frac{\partial}{\partial\lambda}\frac{1}{(1-z)^{\lambda}}=-\frac{1}{(1-z)^{\lambda}}\ln(1-z),$$

and the latter at once yields the required summation (1.1) since

$$\ln(1-z)=-z\sum_{n=0}^{\infty}\frac{z^n}{n+1}=-z\sum_{n=0}^{\infty}\frac{(1)_n(1)_n}{(2)_n}\frac{z^n}{n!}=-z\,{}_2F_1\left[\begin{matrix}1,\,1;\\ 2;\end{matrix}\,z\right]\qquad(|z|<1).$$

Next, we give three short proofs of formula (1.2). First, it is clear that (1.2) is a special case of (1.5), and starting from (1.5) with $\alpha=\lambda$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,z^n&=\frac{z}{\lambda(1-z)^{\lambda+1}}\,{}_3F_2\left[\begin{matrix}\lambda+1,\ \ \lambda,\ \ \lambda;\\ \lambda+1,\lambda+1;\end{matrix}\,\frac{z}{z-1}\right]\\&=\frac{z}{\lambda(1-z)^{\lambda+1}}\,{}_2F_1\left[\begin{matrix}\lambda,\ \ \lambda;\\ \lambda+1;\end{matrix}\,\frac{z}{z-1}\right]\\&=\frac{z}{\lambda(1-z)}\,{}_2F_1\left[\begin{matrix}1,\ \ \lambda;\\ \lambda+1;\end{matrix}\,z\right],\end{aligned}\tag{2.1}$$

where in (2.1) we apply the transformation given in (2.8) below.

Alternatively, if we recall the known functional equation for the digamma function ([1, p 258, equation (6.3.6)] and [2, p 14, equation (1.2.7)])

$$\psi(\lambda+n)-\psi(\lambda)=\sum_{k=0}^{n-1}\frac{1}{\lambda+k}=\frac{1}{\lambda}\sum_{k=0}^{n-1}\frac{(\lambda)_k}{(\lambda+1)_k},\tag{2.2}$$

and the elementary double series identity (see, for instance, [16, p 100, equation (2)])

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,m+n),\tag{2.3}$$

then, upon noting that $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$, it is not difficult to deduce the desired formula (1.2) in the following manner:

$$\begin{aligned}\sum_{n=1}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\,z^n &= \sum_{n=0}^{\infty}[\psi(\lambda+n+1)-\psi(\lambda)]\,z^{n+1}\\ &= \sum_{n=0}^{\infty} z\left(\frac{1}{\lambda}\sum_{k=0}^{n}\frac{(\lambda)_k}{(\lambda+1)_k}\right)z^n = \frac{1}{\lambda}z\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_k}{(\lambda+1)_k}z^{n+k}\\ &= \frac{1}{\lambda}z\sum_{n=0}^{\infty}z^n\sum_{k=0}^{\infty}\frac{(\lambda)_k(1)_k}{(\lambda+1)_k}\frac{z^k}{k!} = \frac{1}{\lambda}\frac{z}{1-z}\,{}_2F_1\left[\begin{matrix}1, & \lambda; \\ & \lambda+1;\end{matrix}\; z\right].\end{aligned}$$

Another similar, but somewhat simpler proof of (1.2) is based on (2.2) with $n = 1$:

$$\psi(z+1)-\psi(z) = \frac{1}{z}. \tag{2.4}$$

Upon setting $z = n+\lambda$ in (2.4), multiplying both sides of the so-obtained equation by z^n and summing over n, $n \in \mathbb{N}_0$, we get

$$\sum_{n=0}^{\infty}\psi(\lambda+n+1)z^n = \sum_{n=0}^{\infty}\psi(\lambda+n)z^n + \sum_{n=0}^{\infty}\frac{z^n}{n+\lambda},$$

from which we have

$$\sum_{n=1}^{\infty}\psi(\lambda+n)z^{n-1} = \psi(\lambda) + \sum_{n=1}^{\infty}\psi(\lambda+n)z^n + \sum_{n=0}^{\infty}\frac{z^n}{n+\lambda},$$

so that (1.2) follows since

$$\sum_{n=0}^{\infty}\frac{z^n}{n+\lambda} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\lambda+1)_n}z^n = \frac{1}{\lambda}\,{}_2F_1\left[\begin{matrix}1, & \lambda; \\ & \lambda+1;\end{matrix}\; z\right].$$

In order to prove the summation formula given by (1.3) we shall need the next result in the theory of the ψ function (see, e.g., [4, p 126, entry (6.6.34)] and the references cited there)

$$\psi(x+y)-\psi(x) = -\sum_{k=1}^{\infty}\frac{(-y)_k}{k(x)_k} \qquad (\Re(x+y) > 0;\, x \notin \mathbb{Z}_0^-) \tag{2.5}$$

as well as the simple summation

$$\sum_{n=0}^{\infty}(-n)_k\frac{z^n}{n!} = (-z)^k\,\mathrm{e}^z \qquad (z \in \mathbb{C};\, k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}). \tag{2.6}$$

Observe that the expansion (2.6) is readily available:

$$\sum_{n=0}^{\infty}(-n)_k\frac{z^n}{n!} = (-1)^k\sum_{n=0}^{\infty}\frac{n!}{(n-k)!}\frac{z^n}{n!} = (-1)^k\sum_{n=k}^{\infty}\frac{z^n}{(n-k)!} = (-z)^k\,\mathrm{e}^z$$

with the aid of

$$(-l)_k = (-1)^k\frac{l!}{(l-k)!} \qquad (l \in \mathbb{N}_0;\, k = 0, 1, \ldots, l).$$

Now, in view of (2.5), (2.6) and [15, p 758]

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n \qquad (m, n \in \mathbb{N}), \tag{2.7}$$

it is easy to see that we have

$$\begin{aligned}\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-n)_k}{k(\lambda)_k}\frac{z^n}{n!}\\ &= \sum_{n=0}^{\infty}(-n)_k\frac{z^n}{n!}\sum_{k=1}^{\infty}\frac{1}{k(\lambda)_k} = -\mathrm{e}^z\sum_{k=1}^{\infty}\frac{(-z)^k}{k(\lambda)_k} = z\,\mathrm{e}^z\sum_{k=0}^{\infty}\frac{(-z)^k}{(k+1)(\lambda)_{k+1}}\\ &= \frac{1}{\lambda}z\,\mathrm{e}^z\sum_{k=0}^{\infty}\frac{(1)_k(1)_k}{(2)_k(\lambda+1)_k}\frac{(-z)^k}{k!} = \frac{1}{\lambda}z\,\mathrm{e}^z{}_2F_2\left[\begin{matrix}1, & 1;\\ 2, & \lambda+1;\end{matrix}\; -z\right],\end{aligned}$$

thus in this way we arrive at the proposed formula (1.3).

Further, to obtain equation (1.4), we make use of the Kummer first formula for the confluent hypergeometric function ${}_1F_1$ [15, p 579, equation (7.11.1.2)]

$${}_1F_1\left[\begin{matrix}a;\\ b;\end{matrix}\; z\right] = \mathrm{e}^z{}_1F_1\left[\begin{matrix}b-a;\\ b;\end{matrix}\; -z\right]$$

and the relation (1.7), thus we need only to verify the following straightforward evaluation:

$$\begin{aligned}\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\frac{z^n}{(\lambda)_n} &= -\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{z^n}{(\lambda)_n}\right\} = -\frac{\partial}{\partial\lambda}\left\{{}_1F_1\left[\begin{matrix}1;\\ \lambda;\end{matrix}\; z\right]\right\}\\ &= -\mathrm{e}^z\frac{\partial}{\partial\lambda}\left\{{}_1F_1\left[\begin{matrix}\lambda-1;\\ \lambda;\end{matrix}\; -z\right]\right\} = -\mathrm{e}^z\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{(\lambda-1)_n}{(\lambda)_n}\frac{(-z)^n}{n!}\right\}\\ &= -\mathrm{e}^z\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{\lambda-1}{\lambda-1+n}\frac{(-z)^n}{n!}\right\} = -\mathrm{e}^z\sum_{n=1}^{\infty}\frac{1}{(\lambda-1+n)^2}\frac{(-z)^{n-1}(-z)}{(n-1)!}\\ &= z\,\mathrm{e}^z\sum_{n=0}^{\infty}\frac{1}{(\lambda+n)^2}\frac{(-z)^n}{n!} = \frac{1}{\lambda^2}z\,\mathrm{e}^z{}_2F_2\left[\begin{matrix}\lambda, & \lambda;\\ \lambda+1, & \lambda+1;\end{matrix}\; -z\right].\end{aligned}$$

Similarly, having in mind the well-known transformation of ${}_2F_1$ function [15, p 454, equation (7.3.1.4)]

$${}_2F_1\left[\begin{matrix}a, b;\\ c;\end{matrix}\; z\right] = \frac{1}{(1-z)^a}{}_2F_1\left[\begin{matrix}a, c-b;\\ c;\end{matrix}\; \frac{z}{z-1}\right] \tag{2.8}$$

and the fact that $(\alpha)_n = \alpha(\alpha+1)_{n-1}$ (see equation (2.7)), we derive formula (1.5) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}[\psi(\lambda+n)-\psi(\lambda)]\frac{(\alpha)_n}{(\lambda)_n}z^n &= -\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\lambda)_n}z^n\right\} = -\frac{\partial}{\partial\lambda}\left\{{}_2F_1\left[\begin{matrix}\alpha, 1;\\ \lambda;\end{matrix}\; z\right]\right\}\\ &= -\frac{\partial}{\partial\lambda}\left\{\frac{1}{(1-z)^\alpha}{}_2F_1\left[\begin{matrix}\alpha, \lambda-1;\\ \lambda;\end{matrix}\; \frac{z}{z-1}\right]\right\}\\ &= -\frac{1}{(1-z)^\alpha}\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n(\lambda-1)_n}{(\lambda)_n n!}\left(\frac{z}{z-1}\right)^n\right\}\\ &= -\frac{1}{(1-z)^\alpha}\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)_{n-1}}{(\lambda-1+n)^2}\frac{1}{(n-1)!}\left(\frac{z}{z-1}\right)^n\\ &= \frac{\alpha z}{\lambda^2(1-z)^{\alpha+1}}\sum_{n=0}^{\infty}\frac{(\alpha+1)_n(\lambda)_n(\lambda)_n}{(\lambda+1)_n(\lambda+1)_n n!}\left(\frac{z}{z-1}\right)^n.\end{aligned}$$

Lastly, note that [15, p 594, entry (7.13.1)]

$$ {}_0F_1\left[\begin{matrix}-;\\ a;\end{matrix}\; z\right] = \Gamma(a) z^{\frac{1-a}{2}} I_{a-1}(2\sqrt{z}), $$

$I_\nu(z)$ being the modified Bessel function [1, p 374, section 9.6], so that, by (see equation (1.7))

$$ \sum_{n=0}^{\infty} [\psi(\lambda+n) - \psi(\lambda)] \frac{z^n}{(\lambda)_n n!} = -\frac{\partial}{\partial\lambda}\left\{\sum_{n=0}^{\infty}\frac{1}{(\lambda)_n}\frac{z^n}{n!}\right\} = -\frac{\partial}{\partial\lambda}\left\{{}_0F_1\left[\begin{matrix}-;\\ \lambda;\end{matrix}\; z\right]\right\}, $$

we have (1.6).

3. Concluding remarks

In conclusion, in this work we have given simple direct proofs of several (new and known) summation formulae: formula (1.5) is new, Miller recently deduced (1.2) and (1.3) [9, p 3015, equation (3.6) and p 3014, equation (3.3)], formula (1.4) is essentially related to the Miller result [9, p 3011, equation (1.1b)]] while (1.1) and (1.6) are long well known [4, p 363, equations (55.7.5) and (55.7.11)].

It should be remarked that all summations (1.1)–(1.6) are, in fact, special cases of one (the most) general which involves the Kampé de Fériet double hypergeometric function in two variables $F^{\text{p:r;u}}_{\text{q:s;v}}$ (see [12] for an introduction to these functions). Indeed, the series (1.1)–(1.6) are clearly of the form (3.1), then by proceeding as in the second proof of (1.2), i.e. by using (2.2) and (2.3) as well as (2.7), we have (cf [9, p 3016, equation (5.1)])

$$ \sum_{n=1}^{\infty} [\psi(\lambda+n) - \psi(\lambda)] \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} z^n \tag{3.1} $$

$$ \begin{aligned} &= \sum_{n=0}^{\infty} [\psi(\lambda+n+1) - \psi(\lambda)] \frac{(\alpha_1)_{n+1} \cdots (\alpha_p)_{n+1}}{(\beta_1)_{n+1} \cdots (\beta_q)_{n+1}} z^{n+1} \\ &= z\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\lambda)_k}{(\lambda+1)_k}\frac{(\alpha_1+1)_n\cdots(\alpha_p+1)_n}{(\beta_1+1)_n\cdots(\beta_q+1)_n}z^n \\ &= z\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q}\frac{1}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_k}{(\lambda+1)_k}\frac{(\alpha_1+1)_{n+k}\cdots(\alpha_p+1)_{n+k}}{(\beta_1+1)_{n+k}\cdots(\beta_q+1)_{n+k}}z^{n+k} \\ &= \frac{z}{\lambda}\frac{\alpha_1\cdots\alpha_p}{\beta_1\cdots\beta_q} F^{\text{p:2;1}}_{\text{q:1;0}}\left[\begin{matrix}\alpha_1+1,\ldots,\alpha_p+1 : 1, & \lambda; & 1;\\ \beta_1+1,\ldots,\beta_q+1 : & \lambda+1; & -;\end{matrix}\; z, z\right]. \end{aligned} \tag{3.2} $$

Finally, we may use equations (1.5) and (3.2) to obtain a new reduction formula for the Kampé de Fériet function $F^{1:2;1}_{1:1;0}[z, z]$:

$$ F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha : 1, \beta-1; & 1;\\ \beta : \quad \beta; & -;\end{matrix}\; z, z\right] = \frac{1}{(1-z)^{\alpha}}\,{}_3F_2\left[\begin{matrix}\alpha, \beta-1, \beta-1; \\ \beta, \quad \beta;\end{matrix}\; \frac{z}{z-1}\right] \qquad (|z| < 1). \tag{3.3} $$

Acknowledgments

The author would like to thank Professor Khristo N Boyadzhiev (Ohio Northern University) for his help and valuable comments and suggestions during preparation of this manuscript.

Constructive and valuable comments provided by an anonymous editorial board member of this journal are greatly acknowledged and appreciated. The present investigation was supported by the *Ministry of Science of the Republic of Serbia* under research project number 144004.

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